J. Appl. Maths Mechs Vol. 56, No. 3, pp. 390–395, 1992
Printed in Great Britain.

0021–8928/92 \$15.00+.00

FREE HIGH-FREQUENCY VIBRATIONS OF ANISOTROPIC PLATES OF VARIABLE THICKNESS†

P. E. TOVSTIK

St Petersburg

(*Received* 15 *July* 1991)

Part of the spectrum of the frequencies of free high-frequency vibrations of a uniform anisotropic plate of variable thickness is investigated by the method of asymptotic integration of the three-dimensional dynamic equations of the theory of elasticity. The modes of vibration, which have one or several deformation half-waves in the direction of the plate thickness, are considered. It is assumed that one of the end surfaces of the plate is plane while the other is smooth with a maximum point. The conditions for the existence of modes of vibration localized in the neighbourhood of the maximum point of the plate thickness are found, and approximate expressions are obtained for the frequencies and modes of vibration of this kind.

THE INTEREST in vibrations of this kind is due to quartz resonators whose construction can have the form of plates of variable thickness [1–4], where one of the end surfaces is plane and the other spherical. High-frequency vibrations of shells, which are accompanied by wave formation in the direction of the thickness were investigated in [5].

1. We will write the system of equations of harmonic vibrations of a three-dimensional elastic anisotropic body in a Cartesian system of coordinates x_1, x_2, x_3 [6]

$$E_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l}+\rho\omega^2 u_i=0 \quad (E_{ijkl}=E_{jikl}=E_{klij}) \tag{1.1}$$

where u_k are the projections of the variable, E_{ijkl} are the components of the tensor of the moduli of elasticity, ρ is the density and ω is the frequency of the vibrations. The summation is carried out over repeated subscripts, where the Latin subscripts take the values 1, 2, 3, while the Greek subscripts take the values 1, 2.

Suppose the plate occupies the region

$$0\leqslant x_3\leqslant h(x_1,\ x_2)$$

$$h(x_1,\ x_2)=h_0-{}^1/{}_2R^{-1}f_2+R^{-2}f_3+R^{-3}f_4+\ldots$$

$$f_k(x_1,x_2)=\sum_{i+j=k} d_{ij}x_1{}^i x_2{}^j,\quad k=2,3,\ldots$$

and the thickness of the plate h is a maximum at $x_1 = x_2 = 0$. Here R is the characteristic radius of curvature of the upper end surface and f_k are homogeneous polynomials in x_1 and x_2 of degree k with dimensionless coefficients d_{ij}, and the quadratic form f_2 is assumed to be positive definite.

The end surfaces of the plate are assumed to be free, which gives the following boundary conditions:

$$\begin{gathered}\sigma_{i3}=0\ \ (x_3=0)\\ \sigma_{i3}-(\partial h/\partial x_\alpha)\sigma_{i\alpha}=0\ \ (x_3=h(x_1,\ x_2))\\ \sigma_{ij}=E_{ijkl}\partial u_k/\partial x_l,\ \ i,\ j=1,\ 2,\ 3\end{gathered} \tag{1.2}$$

† *Prikl. Mat. Mekh.* Vol. 56, No. 3, pp. 473–479, 1992.

Below we will consider only modes of vibration that decay exponentially as $x_1^2+x_2^2$ increases and are henceforth called localized. The boundary conditions on the end surface have no effect on these vibrations and will therefore not be specified more exactly.

2. We will assume first of all that the thickness of the plate is constant ($h(x_1, x_2) = h_0$ and the functions u_i depend only on x_3. Then the boundary-value problem (1.1), (1.2) reduces to the form

$$E_{i3k3}\partial^2 u_k/\partial x_3^2+\rho\omega^2 u_i=0, \quad \partial u_k/\partial x_3=0 \quad (x_3=0,\ h_0) \tag{2.1}$$

and defines three series of vibration frequencies

$$\omega_{pn}=n\pi h_0^{-1}(\lambda_p\rho^{-1})^{1/2} \quad (p=1,\ 2,\ 3;\ n=1,\ 2,\dots) \tag{2.2}$$

where λ_p are the eigenvalues of the matrix E_{i3k3}.

The following types of vibrations correspond to the frequencies (2.2):

$$u_i^p(x_3)=U_i^p\cos(n\pi h_0^{-1}x_3) \tag{2.3}$$

where U_i^p are the eigenvectors of the matrix E_{i3k3}, which are assumed to be normalized

$$E_{i3k3}U_k^p=\lambda_p U_i^p, \quad U_i^p U_i^q=\delta_{pq} \tag{2.4}$$

The frequencies of the localized modes of vibration considered below are grouped in the region of the frequencies ω_{pn}.

3. We will carry out a number of transformations. Instead of the unknown functions u_i we will introduce the functions v_p in accordance with the formulas

$$u_i(x_1,\ x_2,\ x_3)=U_i^p v_p(x_1,\ x_2,\ x_3) \tag{3.1}$$

Then relations (1.1) and (1.2), after multiplying by U_i^q and adding, take the form

$$C_{qjpl}\partial^2 v_p/\partial x_j\partial x_l+\rho\omega^2 v_q=0, \quad C_{qjpl}=E_{ijkl}U_i^q U_k^p \tag{3.2}$$

$$C_{q3pl}\partial v_p/\partial x_l-(\partial h/\partial x_\alpha)C_{q\alpha pl}\partial v_p/\partial x_l=0 \quad \text{for} \quad x_3=h(x_1,\ x_2) \tag{3.3}$$

For $x_3 = 0$ the boundary condition is obtained by omitting the second term on the left-hand side of (3.3).

We will introduce the small parameter μ and extend the scales of the variables x_i using the formulas

$$\begin{gathered}\mu^4=h_0R^{-1},\ x_\alpha=R\mu^3y_\alpha \quad (\alpha=1,\ 2),\ x_3=h_0fz\\ f=f(y_\alpha,\ \mu)=1-{}^1/_2\mu^2f_2(y_\alpha)+\mu^5f_5(y_\alpha)+\dots\end{gathered} \tag{3.4}$$

Then $0\leqslant z\leqslant 1$ and, as later calculations show, in the region of interest $y_\alpha\sim 1$.

Instead of the unknown functions $v_i(x_j)$ we will introduce the function

$$v_i'(y_\alpha,\ z)\equiv v_i(x_\alpha,\ x_3) \tag{3.5}$$

Then

$$\begin{gathered}\partial v_i/\partial x_\alpha=h_0^{-1}\mu D_\alpha v_i',\ \partial v_i/\partial x_3=(h_0f)^{-1}\partial v_i'/\partial z\\ D_\alpha=\partial/\partial y_\alpha-zf^{-1}(\partial f/\partial y_\alpha)\partial/\partial z,\ \alpha=1,\ 2\end{gathered} \tag{3.6}$$

By virtue of the fact that, by (3.4), $\partial f/\partial y_\alpha = O(\mu^2)$, the differential operators and $\partial/\partial y_\alpha$ differ only in small terms. Henceforth, for brevity, we will omit the prime on v_i.

After making the changes (3.4)–(3.6), Eqs (3.2) and the boundary conditions (3.3) take the form

$$\begin{gathered}\lambda_qf^{-2}\partial^2v_q/\partial z^2+h_0^2\rho\omega^2v_q+\mu B_{q\alpha p}D_\alpha(f^{-1}\partial v_p/\partial z)+\\ +\mu^2C_{q\alpha p\beta}D_\alpha D_\beta v_p=0,\ B_{q\alpha p}=C_{q\alpha p3}+C_{q3p\alpha}\end{gathered} \tag{3.7}$$

$$f^{-1}\lambda_q \partial v_q/\partial z + \mu C_{q3p\alpha} D_\alpha v_p - \\ -\mu(\partial f/\partial y_\alpha) + \mu(C_{q\alpha p3} f^{-1} \partial v_p/\partial z + C_{q\alpha p\beta} D_\beta v_p) = 0 \quad (z=1) \tag{3.8}$$

The boundary conditions for $z = 0$ are obtained from (3.8) by dropping terms with the factor $\partial f/\partial y_\alpha$. Note that these terms are of the order of μ^3 compared with the principal terms and have no effect when constructing the first approximations obtained below. This fact enables us to extend the results obtained to plates both of whose end surfaces are not plane

$$h_1(x_\alpha) \leqslant x_3 \leqslant h_2(x_\alpha), \quad h(x_\alpha) = h_2 - h_1 \tag{3.9}$$

taking $h(x_\alpha)$ from (3.9) in subsequent calculations.

4. We will choose one of the eigenvalues λ_q of the matrix E_{i3k3} and we will attempt to construct the corresponding three-parameter series of frequencies and modes of localized vibration. Without loss of generality we will assume that $\lambda_q = \lambda_3$. We will also assume that $\lambda_\alpha \neq \lambda_3$ ($\alpha = 1, 2$).

We will seek the solution in the form of formal series of powers of

$$v_q = v_q^{(0)} + \mu v_q^{(1)} + \mu^2 v_q^{(2)} + \ldots, \quad q = 1, 2, 3 \\ v_\alpha^{(0)} = 0, \quad \alpha = 1, 2; \quad v_3^{(0)} \neq 0 \\ \omega = \omega_{3n}(1 + \mu^2\beta_2 + \mu^4\beta_4 + \ldots), \quad n = 1, 2, \ldots \tag{4.1}$$

The unknown functions $v_p^{(i)}(y_\alpha, z)$ and numbers β_k are found by substituting the series (4.1) into (3.7) and (3.8). For μ^0 we obtain the homogeneous boundary-value problem

$$\lambda_3 \partial^2 v_3^{(0)}/\partial z^2 + h_0^2 \rho \omega^2 v_3^{(0)} = 0, \quad \partial v_3^{(0)}/\partial z = 0 \quad (z = 0, 1) \tag{4.2}$$

the solution of which

$$v_3^{(0)}(y_\alpha, z) = V(y_\alpha)\cos(n\pi z), \quad \omega = \omega_{3n}, \quad n = 1, 2, \ldots \tag{4.3}$$

agrees with (2.2) and (2.3), but unlike (2.3) contains the function $V(y_\alpha)$, determined from subsequent approximations.

The functions $v_q^{(1)}$ in (4.1) are found from the non-homogeneous boundary-value problems

$$\lambda_q \partial^2 v_q^{(1)}/\partial z^2 + \lambda_3 (n\pi)^2 v_q^{(1)} - n\pi B_{q\alpha 3}(\partial V/\partial y_\alpha)\sin(n\pi z) = 0 \\ \lambda_q \partial v_q^{(1)}\, \partial z + C_{q33\alpha}(\partial V/\partial y_\alpha)\cos(n\pi z) = 0 \quad (z = 0, 1) \tag{4.4}$$

When $q = \beta = 1, 2$, problems (4.4) have the solution

$$v_\beta^{(1)}(y_\alpha, z) = \frac{B_{\beta\alpha 3}}{n\pi(\lambda_3 - \lambda_\beta)} \frac{\partial V}{\partial y_\alpha} \sin(n\pi z) - \left(\frac{B_{\beta\alpha 3}}{\lambda_3 - \lambda_\beta} + \frac{C_{\beta 33\alpha}}{\lambda_\beta}\right) \frac{\partial V}{\partial y_\alpha} \frac{F_\beta(z)}{k_\beta} \\ k_\beta = n\pi\left(\frac{\lambda_3}{\lambda_\beta}\right)^{1/2} \\ F_\beta(z) = \begin{cases} (\cos {}^1\!/_2 k_\beta)^{-1} \sin\xi, & n = 2k \\ (\sin {}^1\!/_2 k_\beta)^{-1} \cos\xi, & n = 2k+1; \quad \xi = k_\beta(z - {}^1\!/_2) \end{cases} \tag{4.5}$$

We see that the solution (4.5) does not exist if $k_\beta = (2k+1)\pi$ for even n and $k_\beta = 2k\pi$ for odd n, i.e. in cases of "internal resonances", when the frequency ω_{3n} [see (2.2)] is identical with one of the frequencies ω_{3m}, and the evenness of the numbers n and m are different. We can establish from the last relations that the internal resonance also serves as an obstacle for constructing solutions in those cases when the evenness of the numbers n and m are the same. The conditions for an internal resonance to occur can be written in the form

$$\lambda_3 n^2 = \lambda_\beta m^2, \quad \beta = 1, 2, \quad m = 1, 2, \ldots \tag{4.6}$$

where the number n is fixed while m and β are variables. Henceforth we will assume that there is no internal resonance.

For $q = 3$, problem (4.4) is the problem "on the spectrum". We will write it in the form

$$\lambda_3 \partial^2 w/\partial z^2 + \lambda_3 (n\pi)^2 w + g(z) = 0, \quad \lambda_3 \partial w/\partial z + h(z) = 0$$
$$(z = 0, \ 1)$$

Then the condition for it to be compatible has the form

$$\int_0^1 g(z) \cos(n\pi z)\, dz + h(0) - (-1)^n h(1) = 0 \tag{4.7}$$

In view of the fact that condition (4.7) is satisfied and $B_{3\alpha3} = 2C_{333\alpha}$, we obtain

$$v_3^{(1)} = -\frac{B_{3\alpha3}}{2\lambda_3} \frac{\partial V}{\partial y_\alpha} \left(z - \frac{1}{2}\right) \cos(n\pi z) + C v_3^{(0)} \tag{4.8}$$

The solution (4.8) is obtained apart from an arbitrary term and is a general solution of the homogeneous problem. Without loss of generality we will assume $C = 0$.

The functions $v_q^{(2)}$ are found from the boundary-value problems

$$\lambda_q \frac{\partial^2 v_q^{(2)}}{\partial z^2} + \lambda_3 (n\pi)^2 v_q^{(2)} + B_{q\alpha p} \frac{\partial^2 v_p^{(1)}}{\partial y_\alpha \partial z} + C_{q\alpha 3\beta} \frac{\partial^2 v_3^{(0)}}{\partial y_\alpha \partial y_\beta} +$$
$$+ (2\lambda_3 \beta_2 - \lambda_q f_2(y_\alpha)) (n\pi)^2 v_q^{(0)} = 0$$
$$\lambda_q \frac{\partial v_q^{(2)}}{\partial z} + C_{q3p\alpha} \frac{\partial v_p^{(1)}}{\partial y_\alpha} = 0 \quad (z = 0, 1)$$

[$v_3^{(0)}$, $v_p^{(1)}$ are given by (4.3), (4.5) and (4.8)].

With the assumptions made, $v_1^{(2)}$ and $v_2^{(2)}$ are defined uniquely. We will now consider the compatibility condition (4.7) required to construct $v_3^{(2)}$. After simplifications it can be written in the form

$$a_{\alpha\beta} \frac{\partial^2 V}{\partial y_\alpha \partial y_\beta} + \lambda_3 (n\pi)^2 (2\beta_2 - f_2(y_\alpha)) V = 0 \tag{4.9}$$

$$a_{\alpha\beta} = a_{\alpha\beta}^{(0)} + a_{\alpha\beta}^{(\gamma)} F_\gamma(0), \quad \alpha, \beta, \gamma = 1, 2$$

$$a_{\alpha\beta}^{(0)} = C_{3\alpha3\beta} - \frac{B_{3\alpha3} B_{3\beta3}}{4\lambda_3} + \frac{B_{3\alpha\gamma} B_{\gamma\beta3}}{\lambda_3 - \lambda_\gamma}$$

$$a_{\alpha\beta}^{(\gamma)} = -\frac{4\lambda_3}{k_\gamma} \left(\frac{B_{3\alpha\gamma}}{\lambda_\gamma - \lambda_3} + \frac{C_{33\gamma\alpha}}{\lambda_3}\right) \left(\frac{B_{\gamma\beta3}}{\lambda_3 - \lambda_\gamma} + \frac{C_{\gamma33\beta}}{\lambda_\gamma}\right) \tag{4.10}$$

[the function $F_\gamma(z)$ is the same as in (4.5)].

This process of constructing successive approximations can be continued.

5. We will seek solutions of Eq. (4.9), which decay exponentially as $y_1^2 + y_2^2 \to \infty$, and the corresponding values of β_2. In [7, 8] solutions were obtained of equations for which (4.9) is a special case. This enables us to suggest a simpler method of solution.

The quadratic form f_2 is positive definite, while the matrix $A = \{a_{\alpha\beta}\}$ is symmetrical. We obtain an affine transformation of the variables

$$y_\alpha = c_{\alpha\beta} t_\beta, \quad C = \{c_{\alpha\beta}\} \tag{5.1}$$

as a result of which $f_2 = t_1^2 + t_2^2$, while the matrix A becomes diagonal. For this, we initially obtain some transformation $y_\alpha = d_{\alpha\beta} z_\beta$, which converts f_2 into $z_1^2 + z_2^2$ [if $h(x_1, x_2)$ is a surface of rotation of radius R at the vertex, this transformation cannot be carried out]. As a result, the matrix A becomes

$$A' = (D^{-1})^T A D^{-1}, \quad D = \{d_{\alpha\beta}\}$$

We further take an orthogonal transformation of rotation

$$z_\alpha = e_{\alpha\beta} t_\beta, \quad E = \{e_{\alpha\beta}\}, \quad E^{-1} = E^T$$

such that the matrix A'' becomes diagonal

$$A'' = EA'E^T = \mathrm{diag}(a_1, a_2)$$

Then the transformation (5.1) with the matrix $C = DE$ will be the required one, while Eq. (4.9) will take the form

$$a_1 \frac{\partial^2 V}{\partial t_1^2} + a_2 \frac{\partial^2 V}{\partial t_2^2} + \lambda_3 (n\pi)^2 (2\beta_2 - t_1^2 - t_2^2) V = 0 \tag{5.2}$$

Suppose first of all that $a_1 > 0$, $a_2 > 0$. Then the decaying solutions of Eq. (5.2) are the products of parabolic-cylinder functions

$$\begin{gathered} V^{(m_1, m_2)} = H_{m_1}(c_1 t_1) H_{m_2}(c_2 t_2) \exp[-{}^1/_2 (c_1^2 t_1^2 + c_2^2 t_2^2)] \\ m_1, \ m_2 = 0, \ 1, \ 2, \ldots, \quad c_\alpha = [\lambda_3 (n\pi)^2 a_\alpha^{-1}]^{1/4} \end{gathered} \tag{5.3}$$

where

$$\beta_2 = \beta_2^{(m_1, m_2)} = \frac{1}{n\pi} \left[\left(\frac{a_1}{\lambda_3} \right)^{1/2} \left(m_1 + \frac{1}{2} \right) + \left(\frac{a_2}{\lambda_3} \right)^{1/2} \left(m_2 + \frac{1}{2} \right) \right]$$

where $H_m(x)$ are Hermitian polynomials of degree m.

Returning to the initial notation, we will represent the approximate expression for the three-parameter series of frequencies corresponding to localized modes of vibration as

$$\begin{gathered} \omega_{3n}^{(m_1, m_2)} = \frac{n\pi}{h_0} \left(\frac{\lambda_3}{\rho} \right)^{1/2} \left\{ 1 + \frac{1}{n\pi} \left(\frac{h_0}{R} \right)^{1/2} \left[\left(\frac{a_1}{\lambda_3} \right)^{1/2} \left(m_1 + \frac{1}{2} \right) + \right. \right. \\ \left. \left. + \left(\frac{a_2}{\lambda_3} \right)^{1/2} \left(m_2 + \frac{1}{2} \right) \right] + O\left(\frac{h_0}{R} \right) \right\} \\ n = 1, 2, \ldots, \ m_1, \ m_2 = 0, \ 1, \ 2, \ldots \end{gathered}$$

and the characteristic scale r of the modes of vibration in a tranversal direction

$$r = \left[\frac{R h_0^3}{\lambda_3 (n\pi)^2} \max(a_1, a_2) \right]^{1/4}$$

For a given n the least frequency of vibrations is obtained for $m_1 = m_2 = 0$, where the polynomials $H_0(x) \equiv 1$ in (5.3).

When constructing higher approximations we arrive at a non-homogeneous equation, the left-hand side of which is identical with (5.2), while the right-hand side contains the function

$$Q(t_1, t_2) \exp[-{}^1/_2 (c_1^2 t_1^2 + c_2^2 t_2^2)]$$

In this case, as in [8], for even approximations, the evenness of the degree of polynomial Q is the same as the evenness of the degree $m_1 + m_2$ of the polynomial in (5.3). In this connection, from the condition for decaying solutions to exist for the even approximations we obtain β_k in (4.1). For odd approximations $\beta_k = 0$, which was noted in (4.1).

If at least one of the numbers a_1 and a_2 is negative or zero, there are no exponentially decaying solutions of Eq. (5.2) and of system (1.1). Consequently, the necessary condition for such solutions to exist is that the matrix A must be positive definite [see (4.10)]. Note that in view of the presence of the factor $F_\gamma(0)$, the elements of this matrix can take any value from $-\infty$ to $+\infty$. In particular, its positive definiteness will necessarily break down in the neighbourhoods of those internal resonances (4.6) for which the evennesses of the numbers m and n are different. The positive definiteness of the matrix A is a fact that, to a known extent, is random and depends on the elastic moduli E_{ijkl}, the choice of one of the eigenvalues λ_p of the matrix E_{i3k3} and the number n of half-waves in the thickness of the plate.

REFERENCES

1. POZDNYAKOV P. G., (Ed.), *Handbook of Quartz Resonators*. Svyaz, Moscow, 1978.
2. ZELENKA M., *Piezoelectric Resonators and Volume and Surface Acoustic Waves. Materials, Technology, Construction and Applications*. Mir, Moscow, 1990.
3. TIERSTEN H. F., Analysis of trapped energy resonators operating in overtones of coupled thickness shear and thickness twist. *Acoust. Soc. Amer.* **59,** 879–888, 1976.
4. SLAVOV S. H., Modes of vibration, motion induction and resonance interval circular convex AT-cut belled design trapped energy quartz resonators. *Appl. Phys.* **A40,** 59–65, 1986.
5. KAPLUNOV Yu. D., High-frequency stress–strain states of small variability in thin elastic shells. *Izv. Akad. Nauk SSSR. MTT*, No. 5, 147–157, 1990.
6. NOWACKI V., *Theory of Elasticity*. Mir, Moscow, 1975.
7. MASLOV V. P., *The Complex WKB Method in Nonlinear Equations*. Nauka, Moscow, 1977.
8. TOVSTIK P. E., The problem of the local loss of stability of shells. *Vestnik LGU* **13,** 72–78, 1982.

Translated by R.C.G.

J. Appl. Maths Mechs Vol. 56, No. 3, pp. 395–402, 1992
Printed in Great Britain.
0021–8928/92 $15.00 + .00

HEAT TRANSFER THROUGH A RIGID DISC PRESSED INTO AN ELASTIC HALF-SPACE†

V. P. LEVITSKII and V. M. ONYSHKEVICH

L'vov

(*Received* 22 *August* 1991)

The axisymmetrical contact problem of the indentation of a rigid disc, modelled by a cylindrical punch, into an elastic half-space is considered. The upper end of the cylinder is subjected to convective heating or cooling and the thermal contact between the punch and the half-space is non-ideal. Outside the region of contact heat exchange occurs with the external medium in accordance with Newton's law. The solution of the thermo-elasticity problem for the half-space is constructed using the Hankel transformation, and the problem of heat conduction for a cylinder is solved by the method of straight lines. The existence of zones where the half-space becomes detached from the punch is established. The temperature fields, heat fluxes and contact stresses in the interacting bodies are found.

1. FORMULATION OF THE PROBLEM

WHEN solving contact problems of thermo-elasticity it is of interest to investigate the phenomenon in which a punch becomes separated from the base [1–3]. However, in these and other

†*Prikl. Mat. Mekh.* Vol. 56, No. 3, pp. 480–486, 1992.